

Planck Distribution in Noncommutative Space

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Abstract

In this study, we derive the Planck distribution function in noncommutative space. It is found that it is modified by a small quantity. It is shown that it is reduced to the usual Planck distribution function in the commutative limit .

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I. INTRODUCTION

In quantum mechanics, the phase space is defined by replacing the canonical position and momentum variables with the Hermitian operators which obey the well-known Heisenberg commutation relations $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$. Later, inspired by the quantum mechanics, it was suggested that one could use the idea of space-time noncommutativity at very small length scales to introduce an effective ultraviolet cutoff [1]

$$[x^\mu, x^\nu] = i\Theta^{\mu\nu}, \quad (1)$$

where $\Theta^{\mu\nu}$ is an antisymmetric tensor describing the strength of the noncommutative effects and plays an analogous role to \hbar in usual quantum mechanics. Note that the matrix $\Theta^{\mu\nu}$ is not a tensor since its elements are identical in all reference frames.

Noncommutative field theories are constructed from the standard field theories just by replacing the usual multiplication with the Moyal \star -product in the Lagrangian density. The \star -product of the two fields up to first order is defined by

$$(\Psi \star \Phi)(x) = \Psi(x)\Phi(x) + \frac{i}{2}\Theta^{\mu\nu}\partial_\mu\Psi(x)\partial_\nu\Phi(x). \quad (2)$$

Now, let us write Hamiltonian of the particle interacting with the radiation field in noncommutative space. The interaction Hamiltonian in noncommutative space is assumed to be obtainable from the standard prescription if we replace the ordinary product with the \star -product. The idea is to formulate the theories in noncommutative space as theories in commutative space and to express the noncommutativity by an appropriate \star -product. Then, the interaction Hamiltonian in noncommutative space is given by

$$\hat{H}_{int.} = -\frac{e}{mc}\hat{\mathbf{A}} \star \mathbf{p}. \quad (3)$$

Note also that $\hat{\mathbf{A}}$ in the above Hamiltonian is the noncommutative gauge field operator. Here, caret indicates the noncommutative quantities.

Gauge theory was formulated in noncommutative space [2–4]. The Seiberg-Witten map

allows us to write the noncommutative fields in terms of the ordinary fields. The Seiberg-Witten map for the gauge field is known to lowest order in $\Theta^{\mu\nu}$ [5]

$$\hat{A}_\nu = A_\nu - \frac{1}{2}\Theta^{\alpha\beta}A_\alpha(\partial_\beta A_\nu + F_{\beta\nu}) . \quad (4)$$

Substitution of the equation (4) into the Hamiltonian (3) and applying the definition (2) yields

$$\hat{H}_{int.} = -\frac{e}{mc}\mathbf{A}\cdot\mathbf{p} - \frac{ie}{2mc}\Theta^{\mu\nu}\partial_\mu\mathbf{A}\cdot\partial_\nu\mathbf{p} + \frac{e}{2mc}\Theta^{\mu\nu}\sum_{j=1}^3 A_\mu(\partial_\nu A_j + F_{\nu j})p_j , \quad (5)$$

where p_j is the j -th component of the momentum operator and $A_j = (0, -\mathbf{A})$. The first term in the above Hamiltonian is the standard one and the other two terms are due to the noncommutative nature of the space.

A few more words about the contribution to the interaction Hamiltonian are in order. Since \mathbf{A} contains only the operators a^\dagger and a for photons linearly, the second term in Eq. (5) induces transitions for which one photon is either produced or annihilated. On the other side, the last term in the equation (5) induces transitions where two photons are involved. Here, we are interested only in one photon processes, so we restrict our consideration to the interaction Hamiltonian which includes \mathbf{A} linearly. So, we omit the last term in Eq. (5).

If we use the plane wave expansion of \mathbf{A} with the box normalization condition, the second term in the above Hamiltonian can be rewritten as

$$\frac{ie}{2mc}\Theta^{\mu\nu}\partial_\mu\mathbf{A}\cdot\partial_\nu\mathbf{p} = \frac{ie}{2mc\hbar}\Theta^{\mu\nu}\sum_{k\rho}N_k k_\mu(a_{k\rho}e^{i(k\cdot x - \omega t)} - a_{k\rho}^\dagger e^{-i(k\cdot x - \omega t)})\hat{\epsilon}_{k\rho}\cdot\mathbf{p} p_\nu , \quad (6)$$

where $\hat{\epsilon}_{k\rho}$ are the polarization vectors ($\rho = 1, 2$) and $N_k = \sqrt{\frac{2\pi\hbar c^2}{L^3\omega_k}}$. Here, we used the following identity: $\partial_\nu\mathbf{p} = -\frac{i}{\hbar}p_\nu\mathbf{p}$.

Having obtained the contribution to the interaction Hamiltonian, now we can calculate the contribution of this noncommutative term to the first order transition matrix element.

$$\langle f, n_{k\rho} \pm 1 | \frac{ie}{2mc\hbar}\Theta^{\mu\nu}\sum_{k\rho}N_k k_\mu(a_{k\rho}e^{i(k\cdot x - \omega t)} - a_{k\rho}^\dagger e^{-i(k\cdot x - \omega t)})\hat{\epsilon}_{k\rho}\cdot\mathbf{p} p_\nu | i, n_{k\rho} \rangle \quad (7)$$

An atom is in the initial state $|i\rangle$ and decays into the final state $|f\rangle$ by emitting a photon with the wave vector \mathbf{k} and polarization ρ . It can also change its state by absorbing a photon with energy $\hbar\omega$. The initial and the final of the total atom+radiation field are denoted by $|i, n_{k\rho}\rangle$ and $|f, n_{k\rho} \pm 1\rangle$, respectively. The number of photons is increased or decreased by one unit. Note that there is a minus sign in front of the operator a^\dagger . This is because of the existence of the derivative of the gauge field operator in Hamiltonian (5). This leads to some differences between the two theories. For example, the transition matrix elements in electric dipole approximation are the same both for absorption and emission processes in usual quantum mechanics. However, they are not equal in noncommutative theories.

Let us calculate the first order transition matrix element for emission of a photon.

$$\frac{e}{mc} \sqrt{n_{k\rho} + 1} N_k \langle f | e^{-ik \cdot x} \hat{\epsilon}_{k'\rho'} \cdot \mathbf{p} \left(1 + \frac{i}{2\hbar} \Theta^{\mu\nu} k_\mu p_\nu\right) | i \rangle . \quad (8)$$

Similarly, the transition matrix element for the absorption of a photon is given by

$$\frac{e}{mc} \sqrt{n_{k\rho}} N_k \langle f | e^{ik \cdot x} \hat{\epsilon}_{k'\rho'} \cdot \mathbf{p} \left(1 - \frac{i}{2\hbar} \Theta^{\mu\nu} k_\mu p_\nu\right) | i \rangle . \quad (9)$$

II. NONCOMMUTATIVE PLANCK DISTRIBUTION

In this section, we derive Planck's radiation formula. It was first derived by Einstein in 1917. We consider a sample of atoms in thermal equilibrium. The number of atoms in state $|f\rangle$ is denoted by N_f and for those in state $|i\rangle$ the number N_i . Transitions occur between the two states; photons absorbed or emitted from the radiation field.

Equilibrium requires that time derivation of the number of atoms is zero.

$$\dot{N}_f = \dot{N}_i = 0 , \quad (10)$$

The number of particles with energy E , in thermal equilibrium at temperature T , is proportional to the Boltzmann factor, so

$$\frac{N_f}{N_i} = \exp\left(-\frac{E_f - E_i}{k_B T}\right) , \quad (11)$$

where T represents temperature and k_B is Boltzmann's constant.

We know from quantum statistical mechanics that this ratio is also equal to [6]

$$\frac{N_f}{N_i} = \frac{(Trans. prob. / time)_{emis.}}{(Trans. prob. / time)_{abs.}}. \quad (12)$$

Now, we are in a position to calculate this ratio by taking the absolute square of the transition matrix elements Eqs.(8,9).

$$\frac{(Trans. prob. / time)_{emis.}}{(Trans. prob. / time)_{abs.}} = \frac{n_{k\rho} + 1}{n_{k\rho}} \times \frac{|\langle f | e^{-ik \cdot x} \hat{\epsilon}_{k\rho} \cdot \mathbf{p} | i \rangle + (i/2\hbar) \Theta^{\mu\nu} k_\mu \langle f | e^{-ik \cdot x} \hat{\epsilon}_{k\rho} \cdot \mathbf{p} p_\nu | i \rangle|^2}{|\langle f | e^{ik \cdot x} \hat{\epsilon}_{k\rho} \cdot \mathbf{p} | i \rangle - (i/2\hbar) \Theta^{\mu\nu} k_\mu \langle f | e^{ik \cdot x} \hat{\epsilon}_{k\rho} \cdot \mathbf{p} p_\nu | i \rangle|^2}. \quad (13)$$

Let us compute the term with $\Theta^{\mu\nu}$ further by using the electric dipole approximation method ($e^{\pm ikx} \approx 1$).

$$\Theta^{\mu\nu} k_\mu \langle f | \hat{\epsilon}_{k\rho} \cdot \mathbf{p} p_\nu | i \rangle = \Theta^{\mu\nu} k_\mu \sum_l \langle f | \hat{\epsilon}_{k\rho} \cdot \mathbf{p} | l \rangle \langle l | p_\nu | i \rangle, \quad (14)$$

where $p_\nu = (p_0, -\mathbf{p})$.

If $\nu = 0$, then $\langle l | p_0 | i \rangle = \frac{E_i}{c} \delta_{li}$. If $\nu \neq 0$, then $\langle l | p_j | i \rangle = im\omega_{li} \langle l | x_j | i \rangle$, where $j = 1, 2, 3$. Then, the equation (14) can be rewritten

$$\frac{im\omega_{fi} E_i}{c} \Theta^{\mu 0} k_\mu \langle f | \hat{\epsilon}_{k\rho} \cdot \mathbf{x} | i \rangle - m^2 \sum_l \Theta^{\mu j} k_\mu \omega_{fl} \omega_{li} \langle f | \hat{\epsilon}_{k\rho} \cdot \mathbf{x} | l \rangle \langle l | x_j | i \rangle. \quad (15)$$

The elements of $\Theta^{\mu\nu}$ are very small [7,8] ($\Theta^{\mu\nu} \ll 1$). So, we can neglect the square terms of $\Theta^{\mu\nu}$. Let us define ϵ which includes $\Theta^{\mu\nu}$ linearly as

$$\epsilon = \frac{\frac{m}{2\hbar} \Theta^{\mu j} k_\mu \sum_l \omega_{fl} \omega_{li} \langle f | \hat{\epsilon}_{k\rho} \cdot \mathbf{x} | l \rangle \langle l | x_j | i \rangle}{\omega_{fi} \langle f | \hat{\epsilon}_{k\rho} \cdot \mathbf{x} | i \rangle}. \quad (16)$$

Then the ratio Eq. (13) is approximated as

$$\frac{(Trans. prob. / time)_{emis.}}{(Trans. prob. / time)_{abs.}} \simeq \frac{n_{k\rho} + 1}{n_{k\rho}} \times \frac{1 - 2\epsilon}{1 + 2\epsilon} \simeq \frac{n_{k\rho} + 1}{n_{k\rho}} \times (1 - 4\epsilon), \quad (17)$$

At the last step, we used the following relation ($\frac{1}{1 + \epsilon} \simeq 1 - \epsilon$). Substitution of the equations (11,17) into the definition (12) leads to the Planck distribution function in noncommutative space.

$$n_{k\rho} = \frac{1}{\kappa \exp(\frac{\hbar\omega_k}{k_B T}) - 1} , \quad (18)$$

where κ is given by

$$\kappa = \frac{1}{1 - 4\epsilon} \simeq 1 + 4\epsilon . \quad (19)$$

In commutative limit ($\Theta^{\mu\nu} \rightarrow 0$, $\kappa \rightarrow 1$), the Eq. (18) is reduced to the usual Planck distribution function.

Cosmic black body radiation may be one of the experimental test of noncommutative theories. The remnants of the intense radiation field produced in the beginning can be presented as a black body radiation. The value of the physical parameters like energy and temperature should be shifted because of the parameter κ in Eq. (18), if noncommutativity of the space plays a role in the first times of big bang. One can study many new interesting physical phenomena with this noncommutative Planck distribution function Eq. (18).

Up to now, we have not said anything about the elements of the noncommutative parameter $\Theta^{\mu\nu}$. Actually, there is no agreement in the literature on how these elements are constructed explicitly. While many researchers set $\Theta^{0i} = 0$ to avoid problems with unitarity and causality [9], the others assume that the components of $\Theta^{\mu\nu}$ are constant over cosmological scales, in any given frame of reference there is a special noncommutative direction [10]. Hewett-Petriello-Rizzo [11] parametrized them with the three different angles. In [12], $\Theta^{\mu\nu}$ is interpreted as a background B-field.

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